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On the harmonic oscillator and its invariance superalgebras

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Abstract. We determine the largest kinematical *and* dynamical invariance Lie superalgebras characterizing n-dimensional harmonic oscillators when the spin-orbit supersymmetrization procedure is under study. These considerations essentially add new Heisenberg superstructures in the case of an arbitrary even number of spatial dimensions.

1. Introduction

Many recent works (Bagchi *et al* 1990) have already been devoted to supersymmetric quantum mechanics as introduced by Witten (Witten 1981). Most of them have put the accent on the supersymmetrized harmonic oscillator (Beckers and Hussin 1986, Beckers *et al* 1987, 1988), an interesting $N=2$ case in connection with *exact* supersymmetry for example. In particular, the conformal invariances (Fubini and Rabinovici 1984) associated with this system have been intensively exploited (Beckers and Hussin 1986, Beckers *et al* 1987, 1988) essentially within the so-called standard procedure of supersymmetrization.

Let us recall that, in this case, the n-dimensional supersymmetric Hamiltonian is defined as the anticommutator of the two following supercharges:

$$Q_{\pm}=\frac{1}{\sqrt{2}}(p_j \mp \mathrm{i}\omega x_j)\xi_{\pm,j} \qquad j=1,\ldots,n \tag{1.1}$$

where summation over repeated indices is understood. The bosonic variables p_j and x_j satisfy as usual

$$[p_j, x_l]=-\mathrm{i}\delta_{jl} \qquad p_j=-\mathrm{i}\frac{\partial}{\partial x_j}. \tag{1.2}$$

Moreover the fermionic quantities $\xi_{\pm,j}$ form the basis of a Clifford algebra $\mathscr{C}l_{2n}$ (Sattinger and Weaver 1986), i.e.

$$\{\xi_{+,j}, \xi_{-,l}\}=\delta_{jl}I \tag{1.3a}$$

$$\{\xi_{\pm,j}, \xi_{\pm,l}\}=0. \tag{1.3b}$$

The corresponding supersymmetric Hamiltonian is then given by

$$H^{\mathrm{ST}}=\{Q_+, Q_-\}=\tfrac{1}{2}(p^2+\omega^2x^2)+\tfrac{1}{2}\omega[\xi_{+,j}, \xi_{-,j}] \tag{1.4}$$

which can be called the standard Hamiltonian.

The largest kinematical and dynamical invariance superalgebras associated with this operator have already been revealed (Beckers *et al* 1988) as $[\mathrm{osp}(2/2)\oplus\mathrm{so}(n)]\,⨭\,\mathrm{sh}(2n/2n)$ and $\mathrm{osp}(2n/2n)\,⨭\,\mathrm{sh}(2n/2n)$ respectively. The symbol ⨭ refers here to a semidirect sum of algebras.

It is also well known that the properties characterizing the fermionic variables are specifically dependent on the supersymmetrization procedure that we want to develop. For example, the conditions (1.3a) can be replaced by

$$\{\xi_{+,j}, \xi_{-,l}\} = \delta_{jl} I - 2i\Xi_{jl} \tag{1.5a}$$

$$\Xi_{jl} = -\Xi_{lj} \qquad \Xi^{\dagger} = \Xi. \tag{1.5b}$$

Taking account of these relations in (1.1), we are led to the following Hamiltonian (Beckers *et al* 1988):

$$H^{SO} = \{Q_+, Q_-\} = \tfrac{1}{2}(p^2 + \omega^2 x^2) + \tfrac{1}{2}\omega[\xi_{+,j}, \xi_{-,j}] - \omega[x_j p_l - x_l p_j]\Xi_{jl} \tag{1.6}$$

differing from the first one (1.4) by a spin–orbit coupling term and thus referring to the so-called spin–orbit supersymmetrization procedure. Let us remember that, in this last context, we know, from the Balantekin contribution (Balantekin 1985), that the three-dimensional supersymmetric oscillator is characterized by osp(2/2)⊕so(3) as the largest dynamical invariance superalgebra. Such a result has also been generalized to an arbitrary number n of spatial dimensions by Kostelecky *et al* (1985). Their study has realized the superalgebra osp(2/2)⊕so(n). More recently, in the two-dimensional case this last structure has been enlarged (Beckers *et al* 1988) to [osp(2/2)⊕so(2)]⊕ sh(2/4) but such an extension has not yet been envisaged when n is greater than two. This is precisely the main purpose of this paper, the extension now being possible due to new properties (Beckers *et al* 1990) that we have obtained on the above *fermionic* variables satisfying (1.5) as typical quantities of the spin–orbit coupling procedure. We have shown that these fermionic variables generate a well-definite semidirect sum of two $\mathscr{C}l_n$ or, equivalently, the unitary superalgebras su($2^{m-1}/2^{m-1}$) and su($2^m/2^m$) in the respective $n=2m$ and $n=2m+1$ cases. These differences between the n-even and n-odd contexts can then be studied at the level of the *invariance* superalgebras associated with the Hamiltonian (1.6). This will thus lead us to distinguish the cases where n is even (section 2) or odd (section 3). Some comments (section 4) on the connection between our results and those obtained in parasupersymmetric quantum mechanics (Beckers and Debergh 1990a, b) will also be presented. These considerations will thus complete the parallelism between the two procedures of supersymmetrization, leading to specific results for kinematical as well as dynamical invariance superstructures.

2. The $n=2m$ case

A fundamental realization (Beckers *et al* 1990) of the unitary superalgebra su($2^{m-1}/2^{m-1}$) generated by the $\xi_{\pm,j}$s ($j=1,\ldots,2m$) is given by ($2^{2m}-1$) matrices of dimension 2^m. Let us also point out that, if we define the new ($4m$) fermionic quantities,

$$\alpha_j = \xi_{+,j} + \xi_{-,j} \qquad \alpha_{n+j} = i(\xi_{-,j} - \xi_{+,j}) \qquad j=1,\ldots,2m \tag{2.1}$$

the relations (1.3b) and (1.5a) now read

$$\{\alpha_j, \alpha_l\} = 2\delta_{jl} I \qquad \{\alpha_{n+j}, \alpha_{n+l}\} = 2\delta_{jl} I \qquad \{\alpha_j, \alpha_{n+l}\} = -\{\alpha_{n+j}, \alpha_l\} = 4\Xi_{jl}. \tag{2.2a, b}$$

We notice that the α_js ($j=1,\ldots,2m$) generate a Clifford algebra $\mathscr{C}l_{2m}$ realized in terms of $2^m \times 2^m$ matrices (Sattinger and Weaver 1986). We can then define

$$\alpha_{n+2j} = -\alpha_{2j-1} \qquad \alpha_{n+2j-1} = \alpha_{2j} \tag{2.3}$$

or equivalently

$$\xi_{+,2j} = -i\xi_{+,2j-1} \qquad \xi_{-,2j} = \mathrm{i}\xi_{-,2j-1} \qquad \forall j = 1, \ldots, 2m. \tag{2.4}$$

Replacing these conditions in (2.2a) and (2.2b), we obtain the results

$$\Xi_{jj+1} = -I \qquad j = 2p+1 \qquad p = 0, \ldots, m-1 \tag{2.5a}$$

$$\Xi_{jl} = 0 \qquad l \neq j+1;\ l = j+1 \qquad j = 2p \qquad p = 1, \ldots, m-1. \tag{2.5b}$$

Taking account of these relations in (1.6), we obtain

$$H^{\mathrm{SO}} = \tfrac{1}{2}(\boldsymbol{p}^2 + \omega^2 \boldsymbol{x}^2) + \omega[\xi_{+,2j-1}, \xi_{-,2j-1}] + \omega(x_j p_{j+1} - x_{j+1} p_j) I. \tag{2.6}$$

It is now easy to see that the usual bosonic operators

$$P_j^{\pm} = \mathrm{e}^{\pm 2\mathrm{i}\omega t}(p_{2j-1} - \omega x_{2j} \mp \mathrm{i} p_{2j} \mp \mathrm{i}\omega x_{2j-1}) \tag{2.7a}$$

$$M_j = p_{2j-1} + \omega x_{2j} \tag{2.7b}$$

$$N_j = p_{2j} - \omega x_{2j-1} \tag{2.7c}$$

and fermionic operators

$$T_j^{\pm} = \mathrm{e}^{\pm 2\mathrm{i}\omega t}(\alpha_{2j-1} \mp \mathrm{i}\alpha_{2j}) \tag{2.8}$$

correspond to effective supersymmetries for the Hamiltonian (2.6). These generators lead to the Heisenberg superalgebra sh($2m/4m$) as a fundamental invariance superalgebra for the ($n = 2m$) harmonic oscillator. Indeed we have on the one hand

$$\begin{aligned} &[H^{\mathrm{SO}}, x_{2j}] = -\mathrm{i}p_{2j} - \mathrm{i}\omega x_{2j-1} \qquad &&[H^{\mathrm{SO}}, p_{2j}] = -\mathrm{i}\omega p_{2j-1} + \mathrm{i}\omega^2 x_{2j} \\ &[H^{\mathrm{SO}}, x_{2j-1}] = -\mathrm{i}p_{2j-1} + \mathrm{i}\omega x_{2j} \qquad &&[H^{\mathrm{SO}}, p_{2j-1}] = \mathrm{i}\omega p_{2j} + \mathrm{i}\omega^2 x_{2j-1} \end{aligned} \tag{2.9}$$

and on the other hand

$$[H^{\mathrm{SO}}, \alpha_{2l}] = -2\mathrm{i}\omega\alpha_{2l-1} \qquad [H^{\mathrm{SO}}, \alpha_{2l-1}] = 2\mathrm{i}\omega\alpha_{2l} \tag{2.10}$$

where we have used the identity

$$[[A, B], C] = \{A, \{B, C\}\} - \{B, \{A, C\}\}. \tag{2.11}$$

Combined with the already known (Kostelecky *et al* 1985) invariance superalgebra osp(2/2)⊕so($2m$), these results lead to the *largest* kinematical invariance superstructure:

$$[\mathrm{osp}(2/2) \oplus \mathrm{so}(2m)] \,\tilde{\oplus}\, \mathrm{sh}(2m/4m). \tag{2.12}$$

Let us notice here that the Kostelecky realization (Kostelecky *et al* 1985)

$$\xi_{\pm,j} = \Gamma_j \otimes \sigma_{\pm} \qquad \{\Gamma_j, \Gamma_l\} = 2\delta_{jl} I \tag{2.13}$$

reduces this last superalgebra to osp(2/2)⊕so($2m$) only. Indeed, this realization, whose matricial dimension is 2^{m+1}, points out non-trivial Ξ_{jl}s, as easily verified.

Let us end this section by noticing that these largest kinematical invariance superalgebras lead to new dynamical invariance superstructures for the ($n = 2m$)-dimensional supersymmetric harmonic oscillator. More precisely, this even case permits the enlargement of the already known structure osp(2/2)⊕so($2m$) (Kostelecky *et al* 1985) to

$$\mathrm{osp}(2m/4m) \,\tilde{\oplus}\, \mathrm{sh}(2m/4m). \tag{2.14}$$

This last superalgebra is generated by the operators defined in (2.7) and (2.8) while

$$C^{\pm}_{jl} = -2\omega(p_j \mp i\omega x_j)(p_l \mp i\omega x_l) \tag{2.15a}$$

$$H_{jl} = 2\omega(p_j - i\omega x_j)(p_l + i\omega x_l) \tag{2.15b}$$

form the simple Lie algebra sp($4m$, ℝ) and

$$J_{jl} = \alpha_j \alpha_l \qquad j, l = 1, \ldots, 2m \tag{2.16}$$

the compact structure so($2m$).

Finally, the odd part of osp($2m/4m$) is given by the products $P_j^+ T_l^{\pm}$, $P_j^- T_l^{\pm}$, $M_j T_l^{\pm}$, $N_j T_l^{\pm}$, showing once again the prominent role played by the Heisenberg superalgebra (Beckers and Hussin 1986).

3. The $n = 2m+1$ case

In comparison with the preceding case, we are concerned here with the unitary superalgebra su($2^m/2^m$) (Beckers *et al* 1990) whose fundamental realization is given by ($2^{m+1} \times 2^{m+1}$) matrices. However, as the Clifford algebras we are dealing with are generated by ($2^m \times 2^m$) matrices (Sattinger and Weaver 1986), it is impossible to find a simple linear relation that will connect the α_js and the α_{n+j}s. Indeed, a detailed calculation shows that the (odd-dimensional) matrix characterizing such a relation would be necessarily antisymmetric in order to satisfy (2.2b). Moreover, the hermiticity of the generators implies the reality of this matrix. These two properties show that the two Clifford algebras are independent due to the singular character of the matrix.

Consequently, the antisymmetric operators Ξ_{jl} cannot be realized through the identity matrix only and an invariance Heisenberg superalgebra cannot appear. We are thus reduced to the kinematical invariance superstructure already obtained in the Kostelecky realization (Kostelecky *et al* 1985), i.e.

$$\text{osp}(2/2) \oplus \text{so}(2m+1). \tag{3.1}$$

Moreover, this structure coincides with the largest dynamical invariance superalgebra associated with the ($n = 2m+1$) supersymmetric oscillator.

4. Comments

As it was already realized within the standard procedure of supersymmetrization, we have put in evidence the largest invariance structures associated with n-dimensional oscillators when the spin-orbit coupling procedure of supersymmetrization is considered. We have distinguished even and odd numbers of spatial dimensions and obtained the *kinematical* invariance superalgebras [osp(2/2)⊕so(n)]$\oplus$sh($n/2n$) and osp(2/2)⊕so(n) respectively. We also have extended these results to *dynamical* superstructures. The parallelism between the two procedures is now completed.

Also, these considerations are of interest because of their perfect inclusion into parasupersymmetric developments (Beckers and Debergh 1990a, b). A complete specification of the supersymmetric theories is thus required as a particular case of parasupersymmetric quantum mechanics. Indeed, the superposition of (para)bosonic and parafermionic degrees of freedom has been recently studied (Beckers and Debergh 1990a) through the spin-orbit coupling procedure of supersymmetrization, leading to remarkable Lie superalgebras and parasuperalgebras (Beckers and Debergh 1990b). It should be noted here that the adding of antisymmetric tensors in (2.2*b*), for example, is recovered in the parasupersymmetric context but through generalized Clifford algebras. These generators essentially reduce to the identity in the even case and Heisenberg parasuperalgebras are found as invariance structures.

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References

Bagchi B, Lahiri A and Roy P K 1990 *Int. J. Mod. Phys.* A **5** 1383–456

Balantekin A B 1985 *Ann. Phys., NY* **164** 277–87

Beckers J and Debergh N 1990a Lie structures in parasupersymmetric quantum mechanics: II. The spin-orbit coupling supersymmetrization procedure *Preprint* ULg

—— 1990b *J. Phys. A: Math. Gen.* **23** L751S

Beckers J, Debergh N, Hussin V and Sciarrino A 1990 *J. Phys. A: Math. Gen.* **23** 3647

Beckers J, Dehin D and Hussin V 1987 *J. Phys. A: Math. Gen.* **20** 1137–54

—— 1988 *J. Phys. A: Math. Gen.* **21** 651–67

Beckers J and Hussin V 1986 *Phys. Lett.* **118A** 319–21

Fubini S and Rabinovici E 1984 *Nucl. Phys.* B **245** 17–44

Kostelecky V A, Nieto M M and Truax D R 1985 *Phys. Rev.* D **32** 2627–33

Sattinger D H and Weaver O L 1986 Lie groups and algebras with applications to physics, geometry and mechanics (Berlin: Springer)

Witten E 1981 *Nucl. Phys.* B **188** 513–54